ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF THE PROBLEM IN THE THEORY OF ELASTICITY FOR THE SPHERICAL SHELL UNDER UNEVEN LOADINGS

PMM Vol. 32, No. 1, 1968, pp. 159-167

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(*Received July 10, 1967*)

The axisymmetric problem of elastic equilibrium of a spherical shell was examined in papers [1 to 3]. A detailed analysis of the asymptotic behavior of the solution for a shell of small thickness is given in paper [4] in connection with the problem of transformation in the limit from three-dimensional problems of the theory of elasticity to two-dimensional problems. However, the question of behavior of the solution for uneven loadings is not examined in this paper. Some results related to this problem were obtained in paper [5].

An analysis is presented for the asymptotic behavior of the stress-deformation state of a closed spherical shell in case of torsion by forces which are uniformly distributed along parallels, and also in the case of loadings which have weaker singularities. The investigation is carried out on the basis of special formulas for summation of series containing Legendre's functions. The derivation of these formulas is carried out with the degree of generality which is necessary for the analysis of the given problem.

1. Summation formulas. Let us examine a series of the following form

$$K(\theta, \xi) = \sum_{k=0}^{\infty} F(x_k) P_k(\cos\theta) P_k(\cos\xi) \qquad F(z) = \frac{zU(z)}{D(z)} \qquad \begin{pmatrix} x_k = k + 1/2 \\ z = x + iy \end{pmatrix} \tag{1.1}$$

Here $P_k(\mu)$ are Legendre's polynomials, $F(z)$ is an odd meromorphic function.

Let us assume that $U(z)$ and $D(z)$ are such that the following conditions are fulfilled.

1) Functions $U(z)$ and $D(z)$ are entire functions, which have real values for $z = x$.

2) The behavior of $F(z)$ outside the vicinity of poles for sufficiently large values $|z|$ is determined by the inequality

$$|F(z)| < M|z|^p e^{-\alpha|x|} \tag{1.2}$$
$$(M, p, \alpha = \text{const}; \; M > 0, \alpha > 0)$$

3) Function $F(z)$ has a countable set of poles. For the sake of simplicity let us assume that on the real axis there are two n-th order poles z_0 and $-z_0$ (if z_0 coincides with any x_k, the series (1.1) does not contain the corresponding term); all other poles are simple and complex. Those which are in the first quadrant are designated by z_k ($k = 1, 2, \ldots$). Because of property 1) it is apparent that $-z_k$, $\bar{z}_k$ and $-\bar{z}_k$ will also be poles together with z_k.

Series of the type (1.1) describe the stress-deformation state of the spherical shell if the external loading is applied in the form of conditions distributed along the parallel $\theta = \xi$ (through the application of the principle of superposition we may obtain the solution also for any distributed loading). In connection with this, conditions are formulated for the function $F(z)$. It is not possible to investigate the behavior of the stress-deformation state of the shell in the case where the shell thickness tends to approach zero, starting directly with series of the type (1.1). For this purpose summation formulas will be utilized the derivation of which is given below.

1. For the sake of definiteness let us assume that $\theta > \xi$ and let us examine the following integrals

$$J^{\pm} = \frac{1}{2\pi i} \oint\limits_{C^{\pm}} \Phi^{\pm}(z)\, dz, \qquad \Phi^{\pm}(z) = F(z)\, P_{z-1/2}(\cos\xi) Q^{\pm}_{z-1/2}(\cos\theta)$$

Here C^+ and C^- are contours represented in Fig. 1, where ζ_1 and ζ_2 are selected in such a manner that C^+ and C^- do not contain poles of function $F(z)$ and $Q^{\pm}_{z-1/2}(\cos\theta)$. Let us note the following properties of the Legendre function of the first kind $P_{z-1/2}(\cos\xi)$, and the Legendre function of the second kind

$$Q^{\pm}_{z-1/2}(\cos\theta) = Q_{z-1/2}(\cos\theta \mp i0)\cdot$$

1) For $|z| \to \infty$ in the region $\delta \leqslant \theta \leqslant \pi - \delta$, $\delta > 0$, we have asymptotic expressions [6]:

$$P_{z-1/2}(\cos\theta) = \left(\frac{2}{\pi z \sin\theta}\right)^{1/2} \left[\cos\left(z\theta - \frac{\pi}{4}\right) + O(z^{-1})\right] \tag{1.3}$$

$$Q^{\pm}_{z-1/2}(\cos\theta) = \left(\frac{\pi}{2z\sin\theta}\right)^{1/2} [1 + O(z^{-1})] \exp\left[\pm i\left(z\theta + \frac{\pi}{4}\right)\right] \tag{1.4}$$

2) Quantities $Q^{\pm}_{z-1/2}(\cos\theta)$ are meromorphic functions of their own index which have simple poles at points $z = -x_k (x_k = k + 1/2)$, where

$$\operatorname{res} Q^{\pm}_{z-1/2}(\cos\theta) = P_k(\cos\theta) \qquad \text{for} \quad z = -x_k \tag{1.5}$$

This relationship is easy to obtain if the following representations are used:

$$Q^{\pm}_{z-1/2}(\cos\theta) = P_{z-1/2}(\cos\theta)\left[\ln\operatorname{ctg}\frac{\theta}{2} - \gamma - \psi(z + 1/2) \pm i\pi/2\right] +$$

$$+ \frac{\cos\pi z}{\pi} \sum_{l=1}^{\infty} \Gamma(l + 1/2 + z)\,\Gamma(l + 1/2 - z)\, \frac{\sigma(l)}{(l!)^2} \sin^{2l}\frac{\theta}{2} \qquad \sigma(l) = \psi(l + 1) + \gamma$$

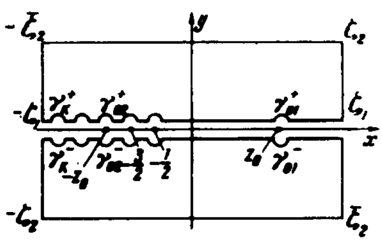

Fig. 1.

Here γ is Euler's constant, $\psi(z + 1/2)$ is the logarithmic derivative of the gamma function. The nonregularity of $Q^{\pm}_{z-1/2}(\cos\theta)$ is connected with the fact that the function $\psi(z + 1/2)$ has simple poles at points $z = -x_k$ $(k = 1, 2, \ldots)$.

Relationship (1.5) is a result of the fact that res $\psi(z + 1/2) = -1$ with respect to $z = -x_k$ and $P_{z-1/2}(\cos\theta) = P_{-z-1/2}(\cos\theta)$.

Now computing J^+ and J^- with the aid of the residue theory, we obtain in the limit for $|\zeta_1|$ and $|\zeta_2| \to \infty$

$$J^+ = \sum_{k=1}^{\infty} \left[\frac{z_k U(z_k)}{D'(z_k)} P_{z_k-1/2}(\cos\xi) Q^{+}_{z_k-1/2}(\cos\theta) + \frac{\bar{z}_k U(\bar{z}_k)}{D'(\bar{z}_k)} P_{\bar{z}_k-1/2}(\cos\xi)\, Q^{+}_{\bar{z}_k-1/2}(\cos\theta)\right] \tag{1.6}$$

$$J^- = \sum_{k=1}^{\infty} \left[\frac{z_k U(z_k)}{D'(z_k)} P_{z_k-1/2}(\cos\xi)\, Q^{-}_{-z_k-1/2}(\cos\theta) + \frac{\bar{z}_k U(\bar{z}_k)}{D'(\bar{z}_k)} P_{\bar{z}_k-1/2}(\cos\xi)\, Q^{-}_{\bar{z}_k-1/2}(\cos\theta)\right] \tag{1.7}$$

Further we represent J^+ and J^- in the form

$$J^+ = \frac{1}{2\pi i} \left\{ \int\limits_{L} - \sum_{k=1}^{[\zeta_1]} \int\limits_{\gamma_k^+} - \int\limits_{\gamma_{01}^+} - \int\limits_{\gamma_{00}^+} + \int\limits_{\zeta_1}^{\zeta_2} + \int\limits_{\zeta_2}^{-\bar{\zeta}_2} + \int\limits_{-\bar{\zeta}_2}^{-\zeta_1} \right\} \Phi^+(z)\, dz \tag{1.8}$$

$$J^- = \frac{1}{2\pi i}\left\{-\int_L - \sum_{k=1}^{[\zeta_1]} \int_{\gamma_k^-} - \int_{\gamma_{01}^-} - \int_{\gamma_{02}^-} + \int_{-\zeta_1}^{-\zeta_2} + \int_{-\bar{\zeta}_2}^{\bar{\zeta}_2} + \int_{\bar{\zeta}_2}^{\zeta_1}\right\} \Phi^-(z)\, dz \tag{1.9}$$

Here L is that part of contours C^+ and C^-, for which points are on the real axis; γ_{01}^+ and γ_{01}^- are half-circles with centers at the point z_0 which are located above and below the real axis, respectively; γ_{02}^+ and γ_{02}^- are half-circles with centers at the point $-z_0$, located above and below the real axis, respectively; γ_k^+ and γ_k^- are half-circles with centers at the points $z = -x_k$, located above and below the half-planes, respectively.

We shall show initially that for $|\zeta_1|$ and $|\zeta_2| \to \infty$ the last three integrals in (1.8) and (1.9) disappear. Let us examine for example the two following integrals:

$$I_1^+ = \int_{\zeta_1}^{\zeta_2} \Phi^+(z)\, dz, \qquad I_2^+ = \int_{\zeta_2}^{-\bar{\zeta}_2} \Phi^+(z)\, dz$$

Changing to integration over y and x, respectively, it is easy to obtain the following estimates with the aid of inequality (1.2) and asymptotic Eqs. (1.3) and (1.4):

$$|I_1^+| < \frac{Me^{-\alpha|x|}}{2\sqrt{\sin\theta\sin\xi}}\left\{\int_0^\infty |z|^{p-1}\,[e^{-(\theta+\xi)y} + e^{-(\theta-\xi)y}]\, dy\right.$$

$$|I_2^+| < \frac{M}{2\sqrt{\sin\theta\sin\xi}}\,[e^{-y(\theta-\xi)} + e^{-(\theta+\xi)y}] \int_{-\infty}^{\infty} |z|^{p-1}\, e^{-\alpha|x|} dx$$

On the basis of assumption $\theta > \xi$, therefore the last inequalities show that I_1^+ and I_2^+ decrease without bound for $|\zeta_1|$ and $|\zeta_2| \to \infty$. We may show in an analogous manner that integrals disappear over the regions $[-\bar{\zeta}_2, \zeta_1]$, $[-\zeta_1, -\zeta_2]$, $[-\zeta_2, \bar{\zeta}_2]$ and $[\bar{\zeta}_2, \zeta_1]$.

Further, examining integrals over γ_k^+ and γ_k^- for $\rho \to 0$ we obtain on the basis of (1.5) and of $F(z)$ being odd;

$$\lim_{\rho\to 0} \frac{1}{2\pi i} \int_{\gamma_k^+} \Phi^+(z)\, dz = \frac{1}{2\pi i} \int_{\gamma_k^-} \Phi^-(z)\, dz = \frac{1}{2} F(x_k)\, P_k(\cos\theta)\, P_k(\cos\xi)$$

After transition to the limit for $|\zeta_1|$ and $|\zeta_2| \to \infty$ for $\rho \to 0$ we arrive at the following representations:

$$J^+ = \frac{K(\theta,\xi)}{2} - \frac{K_0^+(\theta,\xi)}{2} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \Phi^+(x)\, dx$$

$$J^- = \frac{K(\theta,\xi)}{2} - \frac{K_0^-(\theta,\xi)}{2} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \Phi^-(x)\, dx \tag{1.10}$$

Here and below improper integrals are understood in terms of their regular values, $K(\theta, \xi)$ is determined by Eq. (1.1)

$$K_0^{\pm}(\theta,\xi) = \operatorname*{res}_{z=z_0} \Phi^{\pm}(z) + \operatorname*{res}_{z=-z_0} \Phi^{\pm}(z) \tag{1.11}$$

Combining now Eqs. (1.10) and utilizing Expressions (1.6) and (1.7) we obtain the first formula for summation

$$K(\theta,\xi) = K_0(\theta,\xi) + \sum_{k=1}^{\infty}\left\{\frac{z_k U(z_k)}{D'(z_k)} P_{z_k-1/2}(\cos\xi)\,[Q^+_{z_k-1/2}(\cos\theta) + Q^-_{-z_k-1/2}(\cos\theta)] + \right.$$

$$\left. + \frac{\bar{z}_k U(\bar{z}_k)}{D'(\bar{z}_k)} P_{\bar{z}_k-1/2}(\cos\xi)\,[Q^+_{-\bar{z}_k-1/2}(\cos\theta) + Q^-_{\bar{z}_k-1/2}(\cos\theta)]\right\}, \quad K_0(\theta,\xi) = {}^1\!/_2[K_0^+(\theta,\xi) + K_0^-(\theta,\xi)] \tag{1.12}$$

The integral

$$\int_{-\infty}^{\infty} F(x)\, P_{x-1/2}(\cos\xi)\, [Q_{x-1/2}^{+}(\cos\theta) - Q_{x-1/2}^{-}(\cos\theta)]\, dx = 0$$

because $Q_{x-1/2}^{+}(\cos\theta) - Q_{x-1/2}^{-}(\cos\theta) = -i\pi P_{x-1/2}(\cos\theta)$ and $F(x)$ is by assumption an odd function.

2. Let us examine a series with alternating signs of the form

$$K_1(\theta,\ \xi) = \sum_{k=1}^{\infty} (-1)^k F(x_k)\, P_k(\cos\theta)\, P_k(\cos\xi)$$

For the derivation of the summation formula of the series we evaluate the integral

$$J = -\frac{1}{4i} \oint_C \frac{F(z)}{\cos\pi z} P_{z-1/2}(\cos\theta)\, P_{z-1/2}(\cos\xi)\, dz$$

Here C is the contour shown in Fig. 2, where ζ is selected such that the contour does not contain in its vicinity poles of the function under the integral. Since $\cos\pi z$ has zeros at points $z = \pm x_k$ and $F(z)$ is an odd function, we obtain from the residue theorem

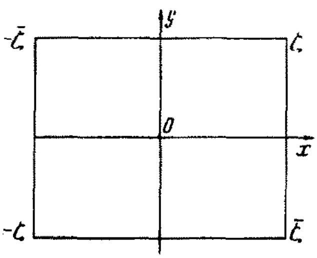

Fig. 2

$$J = \sum_{k=1}^{n} F(x_k)\, P_k(\cos\theta)\, P_k(\cos\xi) + K_0^{(1)} + W_N$$

$$K_0^{(1)} = -\frac{\pi}{2} \sum \operatorname*{res}_{z=\pm z_0} \frac{F(z)}{\cos\pi z} P_{z-1/2}(\cos\theta)\, P_{z-1/2}(\cos\xi)$$

$$W_N = -2\pi \operatorname{Re} \sum_{k=1}^{N} \frac{z_k U(z_k)}{\cos\pi z_k D'(z_k)} P_{z_k-1/2}(\cos\theta)\, P_{z_k-1/2}(\cos\xi)$$

where $n = [\operatorname{Re}\zeta]$, N is the number of poles of the function lying in the first quarter bounded by the rectangle C.

On the basis of inequality (1.2) and asymptotic Eq. (1.3) it is easy to show that for $\theta + \xi < \pi$ and $|\zeta| \to \infty$, the integral approaches 0, therefore after transition to the limit we obtain the following representations for the sum of the series:

$$K_1(\theta,\ \xi) = K_0^{(1)} + W, \qquad W = \lim W_N \quad (N \to \infty) \tag{1.13}$$

2. Torsion of the spherical shell. 1°. Let the closed spherical shell with internal and external radii R_1 and R_2, respectively, be related to a spherical system of coordinates R, θ and ϕ, where θ is the latitude and ϕ the longitude; $\theta = 0$ and $\theta = \pi$ corresponds to poles of the surface R = const.

Let us first examine the case where the shell is deformed under the following conditions

$$\sigma_R = \tau_{R\theta} = \tau_{R\varphi} = 0 \qquad \text{for} \quad R = R_1$$

$$\sigma_R = \tau_{R\theta} = 0, \qquad \tau_{R\varphi} = q\,[\delta(\theta - \xi) - \delta(\theta - \pi + \xi)] \qquad \text{for} \quad R = R_2$$

Here $\delta(\theta)$ is the delta function, q is the magnitude of tangential stresses distributed along parallels $\theta = \xi$ and $\theta = \pi - \xi$.

The solution of this problem obtained by the method of separation of variables has the form

$$u_R = u_\theta = 0, \qquad \sigma_R = \sigma_\theta = \sigma_\varphi = \tau_{R\theta} = 0$$

$$u_\varphi = \frac{q \exp(\lambda + {}^3/_2)\varepsilon}{2G} \sin\xi \sum_{k=1}^{\infty} \frac{x_{2k} U(x_{2k})}{D(x_{2k})} \frac{dP_{2k}(\cos\theta)}{d\theta} \frac{dP_{2k}(\cos\xi)}{d\xi} =$$

$$= \frac{q \exp(\lambda + {}^3/_2)\varepsilon}{4G} \sin\xi \frac{\partial^2}{\partial\theta\, \partial\xi} [K(\theta,\ \xi) + K_1(\theta,\ \xi)] \tag{2.1}$$

Here G is the shear modulus, and $K(\theta, \xi)$ and $K_1(\theta, \xi)$ are determined by Eqs. (1.12) and (1.13):

$$U(z, \lambda) = e^{-3/2\lambda\varepsilon}(2z \operatorname{ch} \lambda\varepsilon z + 3 \operatorname{sh} \lambda\varepsilon z) \qquad (\lambda = 1 / \varepsilon \ln R / R_1) \tag{2.2}$$

$$D(z) = (z^2 - 1/4)(z^2 - 9/4) \operatorname{sh} \varepsilon z \qquad (\varepsilon = \ln R_2 / R_1) \tag{2.3}$$

It is apparent from (2.3) that $D(z)$ has four real zeros $z = \pm 1/2$ and $z = \pm 3/2$ and a countable set of complex $z_k = i(k\pi/\varepsilon)$ $(k = 0, 1, 1, \ldots)$

Stresses $\tau_{\lambda\varphi}$ and $\tau_{\theta\varphi}$ are related to displacements u_φ by Eqs.

$$\tau_{R\varphi} = RG\frac{\partial}{\partial R}\frac{u_\varphi}{R}, \qquad \tau_{\theta\varphi} = \frac{G}{R}\left(\frac{\partial u_\varphi}{\partial\theta} - u_\varphi \operatorname{ctg}\theta\right)$$

Solution (2.1) yields poorly to analysis when $\varepsilon \to 0$ (it is evident from (2.3) that ε characterizes the wall-thinness of the shell). Therefore we shall transform it with the aid of summation Formulas (1.12) and (1.13). We note that the function under the summation sign in (2.1) does not have poles at points $z = \pm 1/2$ because at these values the derivative with respect to θ in (2.1) becomes zero. For u_φ, $\tau_{\lambda\varphi}$ and $\tau_{\theta\varphi}$ we obtain the following Expressions:

$$u_\varphi = u_\varphi^{(0)} + u_\varphi^{(1)}, \qquad \tau_{\theta\varphi} = \tau_{\theta\varphi}^{(0)} + \tau_{\theta\varphi}^{(1)}, \quad \tau_{R\varphi} = \tau_{R\varphi}^{(1)} \tag{2.4}$$

Here

$$u_\varphi^{(0)} = \frac{3qe^{(\lambda+3/2)\varepsilon}\sin\xi}{16G \operatorname{sh}^3/_2\varepsilon} F_u(\theta, \xi), \qquad \tau_{\theta\varphi}^{(0)} = \frac{3qe^{3/2\varepsilon}\sin\xi}{8R_2 \operatorname{sh} {}^3/_2\varepsilon} F_\tau(\theta, \xi) \tag{2.5}$$

$$u_\varphi^{(1)} = \frac{qe^{(\lambda+3/2)\varepsilon}\sin\xi}{4G} T_1(\theta, \xi, \lambda), \qquad \tau_{R\varphi}^{(1)} = \frac{qe^{3/2\varepsilon}\sin\xi}{4R_2}\frac{\partial T_1}{\partial\lambda}$$

$$\tau_{\theta\varphi}^{(1)} = \frac{qe^{3/2\varepsilon}\sin\xi}{4R_2}\left[\frac{\partial T_1}{\partial\theta} - \operatorname{ctg}\theta T_1(\theta, \xi, \lambda)\right] \tag{2.6}$$

$$F_u(\theta, \xi) = 2\sin\xi(\sin\theta \ln \operatorname{ctg} {}^1/_2\theta + \operatorname{ctg}\theta) \tag{2.7}$$

$$F_\tau(\theta, \xi) = -(2\sin\xi/\sin^2\theta) \tag{2.8}$$

Expressions (2.7) and (2.8) are applicable for $\theta > \xi$;

$$T_1(\theta, \xi, \lambda) = \sum_{k=1}^{\infty}\frac{z_k U(z_k)}{(z_k^2 - 1/4)(z_k^2 - 9/4)\varepsilon \operatorname{ch} \varepsilon z_k} P^{(1)}_{-1/2+z_k}(\cos\xi)\,[2Q^{(1)}_{-1/2+z_k}(\cos\theta) +$$
$$+ \pi\frac{\sin\pi z_k + 1}{\cos\pi z_k} P^{(1)}_{-1/2+z_k}(\cos\theta)] \tag{2.9}$$

For $\theta < \xi$ in the expression for $u_\varphi^{(0)}$ it is necessary to replace $F_u(\theta, \xi)$ by $F_u(\xi, \theta)$, in Eqs. (2.6) it is necessary to replace $T_1(\theta, \xi, \lambda)$ by $T_1(\xi, \theta, \lambda)$ and in the equation for $\tau_{\theta\varphi}^{(0)}$ $F_\tau(\theta, \xi)$ by

$$F^{(1)}(\theta, \xi) = 0 \tag{2.10}$$

We shall show that for sufficiently small expressions $u_\varphi^{(1)}$, $\tau_{\theta\varphi}^{(1)}$ and $\tau_{R\varphi}^{(1)}$ are localized in the vicinity of the applied stress $(\theta = \xi, \theta = \pi - \xi)$. For this purpose it is apparently sufficient to show that the function $T_1(\theta, \xi, \lambda)$ decreases rapidly with increasing distance from the indicated parallels.

Let us turn to Expression (2.9). We note that quantities $z_k = ik\pi/\varepsilon$ have large values for small ε. Therefore, substituting in (2.9) $Q^{(1)}_{-1/2+z_k}(\cos\theta)$ and $P^{(1)}_{-1/2+z_k}(\cos\theta)$ by two terms of their asymptotic expansion, we may after some transformations, represent $T_1(\theta, \xi, \lambda)$ in the form

$$T_1(\theta, \xi, \lambda) = e^{-3/2\lambda\varepsilon}[T_1^{(0)}(\theta, \xi, \lambda) + \varepsilon T_1^{(1)}(\theta, \xi, \lambda) + O(\varepsilon^2)] \tag{2.11}$$

Here

$$T_1^{(0)}(\theta,\xi,\lambda)=\frac{2}{\sqrt{\sin\theta\sin\xi}}\sum_{k=1}^{\infty}\frac{(-1)^k\cos\lambda\pi k}{k}N_k^{(0)}(\theta,\xi) \tag{2.12}$$

$$T_1^{(1)}(\theta,\xi,\lambda)=\frac{3}{\pi^2\sqrt{\sin\theta\sin\xi}}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\left[\sin\lambda\pi k N_k^{(0)}+\frac{1}{4}\cos\lambda\pi k N_k^{(1)}\right] \tag{2.13}$$

$$N_k^{(0)}(\theta,\xi)=\exp\left[-\frac{k\pi}{\varepsilon}(\theta-\xi)\right]-\exp\left[-\frac{k\pi}{\varepsilon}(\pi-\xi-\theta)\right]+$$
$$+\operatorname{sh}\frac{k\pi}{\varepsilon}\left(\frac{\pi}{2}-\xi\right)\operatorname{sh}\frac{k\pi}{\varepsilon}\left(\frac{\pi}{2}-\theta\right)\operatorname{sch}\frac{k\pi^2}{\varepsilon}\exp\left(-\frac{k\pi^2}{\varepsilon}\right) \tag{2.14}$$

$$N_k^{(1)}(\theta,\xi)=2\operatorname{sch}\frac{k\pi^2}{\varepsilon}\left[\operatorname{ch}\frac{k\pi}{\varepsilon}\left(\frac{\pi}{2}+\xi\right)\operatorname{ch}\frac{k\pi}{\varepsilon}\left(\frac{\pi}{2}-\theta\right)\operatorname{ctg}\theta-\right.$$
$$\left.-\operatorname{sh}\frac{k\pi}{\varepsilon}\left(\frac{\pi}{2}+\xi\right)\operatorname{sh}\frac{k\pi}{\varepsilon}\left(\frac{\pi}{2}-\theta\right)\operatorname{ctg}\xi\right] \tag{2.15}$$

On the basis of (2.11) to (2.15) we may draw the conclusion that $u_\varphi^{(1)}$, $\tau_{\theta\varphi}^{(1)}$ and $\tau_{R\varphi}^{(1)}$ are solutions of the boundary layer type, localized in the vicinity of the line of stress application. In this case they decay the faster, the smaller the parameter of wall thinness ε. Consequently, at sufficient distance from line $\theta=\xi$ and $\theta=\pi-\xi$, the stress-deformation state of the shell is determined by Expressions $u_\varphi^{(0)}$ and $\tau_{\theta\varphi}^{(0)}$. Let us represent $u_\varphi^{(0)}$ and $\tau_{\theta\varphi}^{(0)}$ in the form

$$u_\varphi^{(0)}=\frac{3q\sin\xi}{4G\varepsilon_0}F_u(\theta,\xi)\left[1+\left(1+\frac{2\eta}{h}\right)\varepsilon_0+\left(\frac{2\eta}{h}+\frac{1}{6}\right)\varepsilon_0^2\right]$$
$$\tau_{\theta\varphi}^{(0)}=-\frac{q\sin^2\xi}{R_0\varepsilon_0}\frac{1}{\sin^2\theta}\left(1+\varepsilon_0-\frac{1}{12}\varepsilon_0^2\right) \tag{2.16}$$

$$\varepsilon_0=h/R_0,\qquad h=R_2-R_1,\qquad R_0={}^1/_2(R_2+R_1),\qquad \eta=R-R_0 \tag{2.17}$$

Here h is the thickness of the shell, R_0 is the radius of the mean surface. The first terms of expansions (2.16) and (2.17) represent the momentless solution of the classic theory of shells. In this manner everywhere in the nonconical sections $\theta=\xi$ and $\theta=\pi-\xi$, the stress-deformation state of the shell coincides with the momentless condition with an accuracy to terms of the order of ε_0

Let us examine now in more detail the behavior of the solution in the vicinity of the line of application of external conditions. Let us consider that sections $\theta=\xi$ and $\theta=\pi-\xi$ are distributed such that the mutual interaction of boundary layers may be neglected, then by virtue of the symmetry of the problem it is sufficient to limit the examination to the stress-deformation state in the vicinity of the conic section $\theta=\xi$. Eliminating in (2.9) the main part and summing it, we obtain for $u_\varphi^{(1)}$, $\tau_{\theta\varphi}^{(1)}$ and $\tau_{R\varphi}^{(1)}$ the following Expressions

$$u_\varphi^{(1)}=-\frac{qe^{1/_2(1-\lambda)\varepsilon}}{4\pi G}\left(\frac{\sin\xi}{\sin\theta}\right)^{1/_2}\left\{\ln 2\left[\operatorname{ch}\frac{\pi}{\varepsilon}(\theta-\xi)+\cos\lambda\pi\right]-\frac{\pi}{\varepsilon}|\theta-\xi|-\right. \tag{2.18}$$
$$\left.-\frac{3\varepsilon}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\left[\sin\lambda\pi k+\frac{\cos\lambda\pi k\operatorname{sign}(\theta-\xi)(\operatorname{ctg}\theta-\operatorname{ctg}\xi)}{4}\exp\left[\frac{-k\pi|\theta-\xi|}{\varepsilon}\right]+O(\varepsilon^2)\right\}$$

$$\tau_{R\varphi}^{(1)}=\frac{qe^{3/_2(1-\lambda)\varepsilon}}{4R_2\varepsilon}\left(\frac{\sin\xi}{\sin\theta}\right)^{1/_2}\left[\frac{\sin\lambda\pi}{\operatorname{ch}[(\pi/\varepsilon)(\theta-\xi)]+\cos\lambda\pi}+\right.$$
$$\left.+\frac{3\varepsilon}{4\pi}\operatorname{sign}(\theta-\xi)(\operatorname{ctg}\xi-\operatorname{ctg}\theta)\operatorname{arctg}\frac{\sin\lambda\pi}{\exp[(\pi/\varepsilon)|\theta-\xi|]+\cos\lambda\pi}+Q(\varepsilon^3)\right] \tag{2.19}$$

$$\tau_{\theta\varphi}^{(1)}=-\frac{qe^{3(1-\lambda)\varepsilon/2}}{4R_2\varepsilon}\left(\frac{\sin\xi}{\sin\theta}\right)^{1/_2}\left\langle\frac{\operatorname{sh}[(\pi/\varepsilon)(\theta-\xi)]}{\operatorname{ch}[(\pi/\varepsilon)(\theta-\xi)]+\cos\lambda\pi}-\operatorname{sign}(\theta-\xi)+\right.$$
$$+\frac{\varepsilon}{\pi}\left\{\frac{1}{8}\varphi(\theta,\xi)\left[\ln 2\left(\operatorname{ch}\frac{\pi}{\varepsilon}(\theta-\xi)+\cos\lambda\pi\right)+\frac{\pi}{\varepsilon}|\theta-\xi|-\right.\right. \tag{2.20}$$

$$- 3 \operatorname{arctg} \frac{\sin \lambda\pi}{\exp[(\pi/\varepsilon)|\theta - \xi|] + \cos\lambda\pi} \Big\} + O(\varepsilon^2) \Big\rangle$$

$\varphi(\theta, \xi) = 3 \operatorname{ctg} \xi - 15 \operatorname{ctg} \theta$ for $\theta > \xi$, $\varphi(\theta, \xi) = -13 \operatorname{ctg} \theta + \operatorname{ctg} \xi$ for $\theta < \xi$

The relationships obtained exhibit the character of singularities which have the displacement u_φ and stresses $\tau_{R\varphi}$ and $\tau_{\theta\varphi}$ in the vicinity of lines of stress application (from Eq. (2.4) it is evident that the penetrating part of solution $u_\varphi^{(0)}$ and $\tau_{\theta\varphi}^{(0)}$ does not have singularities). Namely, for $\theta = \xi$ and $\lambda = 1$ ($R = R_2$), it follows from (2.18), that u_φ has a logarithmic singularity and from (2.19) and (2.20) that $\tau_{\theta\varphi}$ and $\tau_{R\varphi}$ have a power singularity. Consequently, in the vicinity of loading lines the stress-deformation state has three-dimensional character.

It is interesting to note that if the curvature of the shell is permitted to go to zero and θ and ξ are allowed to go to $\frac{1}{2}\pi$, then Eqs. (2.18) to (2.20) in the limit give the solution for a layer on which evenly distributed tangential forces act along a straight line located along the edge.

A comparative analysis of relationships (2.5) and (2.18) to (2.20) shows that in conical sections $\theta = \xi$ and $\theta = \pi - \xi$ $u_\varphi^{(0)}$, $\tau_{\theta\varphi}^{(0)}$ have the order $O(\varepsilon^{-1})$, while the order of $u_\varphi^{(1)}$ and $\tau_{\theta\varphi}^{(1)}$ is $O(1)$ and $O(\varepsilon^{-1})$, respectively. This permits to draw the conclusions that everywhere, with the exception of the line of loading, u_φ for $\varepsilon \to 0$ asymptotically approaches the solution of momentless theory of shells. It also gives for $\tau_{\theta\varphi}$ a correction of the same order of magnitude with respect to ε as $\tau_{\theta\varphi}^{(0)}$ at points of sections mentioned. In this case $\tau_{\theta\varphi}^{(0)}$ and also $\tau_{\theta\varphi}^{(1)}$ have discontinuities of the first kind at $\theta = \xi$ and $\theta = \pi - \xi$, however $\tau_{\theta\varphi} = \tau_{\theta\varphi}^{(0)} + \tau_{\theta\varphi}^{(1)}$ is continuous and has the form

$$\tau_{\theta\varphi} = -\frac{q}{4R_2\varepsilon}\left[1 + \delta(1 - \lambda) + O(\varepsilon)\right]$$

which follows from (2.5), (2.8), (2.9) and (2.20). For $\lambda = 1$, $\theta = \xi$ and $\theta = \pi - \xi$, as was already noted, $\tau_{\theta\varphi}$ has a singularity.

With regard to stress $\tau^{(1)}{}_{R\varphi}$, we note that for $\theta \neq \xi$, $\theta \neq \pi - \xi$ and $\varepsilon \to 0$ it decreases without bound. For $\theta = \xi$:

$$\tau_{R\varphi} = \frac{q}{4R_2\varepsilon}\left[\operatorname{tg}\frac{\lambda\pi}{2} + O(\varepsilon)\right]$$

and as is evident, it has the same order as $\tau_{\theta\varphi}$.

The analysis which was presented shows that in the vicinity of the line of loading in the limit a passage to two-dimensional problems is not possible because the stress condition has an essentially three-dimensional character.

2. Let us examine now the behavior of the stress-deformation state of the shell when the external loading has weaker singularities. We shall assume that the internal surface of the shell is free of stresses and that tangential stresses are given on the external surface

$$\tau_{R\varphi} = q(\theta)$$

With regard to $q(\theta)$ let us assume the following: 1) function $q(\pi - \theta) = -q(\theta)$ 2) function $q(\theta)$ is continuous together with its $n - 1$ derivative, $q^{(n)}(\theta)$ has a discontinuity of the first kind for $\theta = \theta_0$ and $\theta = \pi - \theta_0$, the derivative $q^{(n+1)}(\theta)$ is integrable in the interval $[0, \pi]$.

It is apparent that by virtue of condition 1), the solution of the presented problem is obtained through the application of the principle of superposition to relationships (2.4) to (2.10). In this manner u_φ can be presented in the following form

$$u_\varphi = u_\varphi^{(0)} + u_\varphi^{(1)}$$

$$u_\varphi^{(0)} = \frac{3e^{(\lambda+1/2)\varepsilon}}{16G \operatorname{sh} {}^3/_2\varepsilon}\left[\int_0^\theta F_u(\theta, \xi)\, q(\xi) \sin\xi\, d\xi + \int_\theta^{1/2\pi} F_u(\xi, \theta)\, q(\xi) \sin\xi\, d\xi\right] \tag{2.21}$$

$$u_\varphi^{(1)} = \frac{e^{(\lambda+1/2)\varepsilon}}{4G}\left[\int_0^\theta T_1(\theta, \xi, \lambda)\, q(\xi) \sin\xi\, d\xi + \int_\theta^{1/2\pi} T_1(\xi, \theta, \lambda)\, q(\xi) \sin\xi\, d\xi\right]$$

As a result of some transformations connected with n-fold integration by parts of the right-hand part of (2.21) for $\theta > \theta_0$ $u_\varphi^{(1)}$ can be represented in the form

$$u_\varphi^{(1)} = \frac{e^{(\lambda+1/2)\varepsilon}}{4G}\left\{-2\sum_{j=0}^{p}(-1)^j C_j q_{2j}(\theta) + \sum_{k=1}^{\infty}\frac{z_k U(z_k\lambda)\,[B_{nk}^{(0)} + B_{nk}^{(1)}]}{(z_k^2 - 9/4)(z_k^2 - 1/4)^{p+1}\varepsilon \operatorname{ch} \varepsilon z_k}\right\}$$

Here (2.22)

$$p = \left[\frac{n}{2}\right], \qquad q_{2j} = A^j q, \qquad Aq = \frac{d}{d\theta}\left\{\frac{1}{\sin\theta}\left[\frac{d}{d\theta}(q\sin\theta)\right]\right\}$$

$$C_j = \sum_{k=1}^{\infty}\frac{z_k U(z_k, \lambda)}{(z_k^2 - 9/4)(z_k^2 - 1/4)^{j+1}\varepsilon \operatorname{ch}\varepsilon z_k} = -\operatorname*{res}_{z=1/2} f_j(z) - \operatorname*{res}_{z=3/2} f_j(z)$$

$$f_j(z) = \frac{zU(z, \lambda)}{(z^2 - 9/4)(z^2 - 1/4)^{j+1}\operatorname{sh}\varepsilon z}$$

$$B_k^{(0)} = (-1)^s \sin\theta_0 P_{-1/2+z_k}(\cos\theta_0) S_k^{(1)}(\theta)\,[q^{(n)}(\theta_0 - 0) - q^{(n)}(\theta_0 + 0)] \qquad (n = 2s)$$

(2.23)

$$B_k^{(1)} = (-1)^s\left[S_k^{(1)}(\theta)\int_0^\theta P_{-1/2+z_k}(\cos\xi)\frac{d}{d\xi}(q_n\sin\xi)\,d\xi + P_{-1/2+z_k}(\cos\theta)\times\right.$$

$$\left.\times\int_\theta^{1/2\pi} S_k^{(1)}(\xi)\frac{d}{d\xi}(q_n\sin\xi)\,d\xi\right]$$

$$B_k^{(0)} = \frac{(-1)^{s+1}\sin\theta_0}{z_k^2 - 1/4}P_{-1/2+z_k}^{(1)}(\cos\theta_0) S_k^{(1)}(\theta)\,[q^{(n)}(\theta_0 - 0) - q^{(n)}(\theta_0 + 0)] \qquad (n = 2s + 1)$$

(2.24)

$$B_k^{(1)} = \frac{(-1)^s}{z_k^2 - 1/4}\left[S_k^{(1)}(\theta)\int_0^\theta P_{-1/2+z_k}^{(1)}(\cos\xi)\,q_{n+1}(\xi)\sin\xi\,d\xi + \right.$$

$$\left. + P_{-1/2+z_k}^{(1)}(\cos\theta)\int_\theta^{1/2\pi} S_k^{(1)}(\xi)\,q_{n+1}(\xi)\sin\xi\,d\xi\right]$$

$$S_k^{(1)}(\theta) = 2Q_{-1/2+z_k}^{(1)}(\cos\theta) + \pi\frac{\sin\pi z_k + 1}{\cos\pi z_k}P_{-1/2+z_k}^{(1)}(\cos\theta)$$

It follows from relationships (2.22) to (2.24) that in the vicinity of discontinuities $q^{(n)}(\theta)$ local effects arise of the boundary layer type. However, their effect on all solutions is the weaker, the smoother $q(\theta)$ and the smaller ε. In fact, an analysis of coefficients contained in the second sum (2.22) shows that they have the order $O(\varepsilon^{2p+1})$. Coefficients C_j have the order $O(\varepsilon^{2j+1})$. The first two coefficients have the form:

$$C_0 = -1/2\, e^{-1/2\lambda\varepsilon}\,[(\lambda^2 - 1/8)\varepsilon + 1/2\lambda(\lambda^2 + 1)\varepsilon^2 + 5/12(1/2\lambda^4 - \lambda^3 + 1/8)\varepsilon^3]$$

$$C_1 = 1/768\, e^{-1/2\lambda\varepsilon}(55/2\,\lambda^4 + 20\lambda^2 + 69/5)\varepsilon^3$$

In this manner, even if $q(\theta)$ has a discontinuity of the first kind, Expression $u_\varphi^{(1)}$ has the order $O(\varepsilon)$ at the same time as $u_\varphi^{(0)}$ has the order $O(\varepsilon^{-1})$. The series in (2.22) converges uniformly for any λ, therefore the behavior of u_φ for small ε is determined in the entire region by the first term of expansion $u_\varphi^{(0)}$. This term is the solution of the momentless theory of shells.

The character of behavior of stress $\tau_{\theta\varphi}$ in the vicinity of discontinuity $q(\theta)$ turns out to be more complex. For illustration we shall present the asymptotic equations describing the behavior $\tau_{\theta\varphi}$ in the vicinity of the conic section $\theta = \theta_0$

$$\tau_{\theta\varphi} = \tau_{\theta\varphi}^{(0)} + \tau_{\theta\varphi}^{(1)}$$

Here

$$\tau_{\theta\varphi}^{(0)} = \frac{3e^{3/2\,\varepsilon}}{8R_2 \operatorname{sh} {}^3/_2\varepsilon}\left[\int_0^\theta F_\tau(\theta, \xi)\, q(\xi) \sin\xi\, d\xi + \int_\theta^{1/2\,\pi} F_\tau^{(1)}(\theta, \xi)\, q(\xi) \sin\xi\, d\xi\right]$$

$$\tau_\theta^{(1)} = \frac{1}{8\pi R_2}\left\{\ln 2\left[\operatorname{ch}\frac{\pi}{\varepsilon}(\theta - \theta_0) + \cos\lambda\pi\right] - \frac{\pi}{\varepsilon}|\theta - \theta_0|\right\}[q(\theta_0 - 0) - - q(\theta_0 + 0)] + O(\varepsilon) \tag{2.25}$$

For the discarded terms to have the order indicated, it is sufficient to assume that $q'(\theta)$ at the point $\theta = \theta_0$ has finite limits from the left and from the right side.

It is apparent from relationships (2.25) that $\tau_{\theta\varphi}$ contains in it solutions of the boundary layer type localized in the vicinity of section $\theta = \theta_0$. With respect to ε its order is $O(1)$. It has a logarithmic singularity on the line of discontinuity. Consequently, for $\varepsilon \to 0$, $\tau_{\theta\varphi}$ approaches asymptotically everywhere, with the exception of the line of discontinuity, the value which is determined from the momentless theory of shells.

For the solution of the three-dimensional problem to tend everywhere to the momentless state, it is sufficient for external loading $q(\theta)$ to be continuous.

BIBLIOGRAPHY

1. Galerkin, B.G., Equilibrium of an elastic spherical shell. PMM Vol. 6, No. 6, 1942.
2. Lur'e, A.I., Equilibrium of an elastic symmetrically loaded spherical shell. PMM Vol. 7, No. 6, 1943.
3. Lur'e, A.I., Spatial Problems of the Theory of Elasticity. M., Gostekhizdat, 1955.
4. Vilenskaia, T.V. and Vorovich, I.I., Asymptotic behavior of the solution of the problem of elasticity for a thin spherical shell. PMM Vol. 30, No. 2, 1966.
5. Ustinov, Iu.A., Transition from the three-dimensional problem of the theory of elasticity to the two-dimensional for a closed spherical shell under uneven external loading. Tr. VI Vsesoiuz. konfer. po teorii plastin i obolochek. M., Izd. "Nauka", 1965.
6. Bateman, G. and Erdin, A., Higher Transcendental Functions. Vol. 1. M., Izd. "Nauka", 1965,

Translated by B.D.